

On Partitioning the Edges of 1-Plane Graphs

William J. Lenhart^a, Giuseppe Liotta^b, Fabrizio Montecchiani^{b,*}

^aDepartment of Computer Science, Williams College, USA

^bDipartimento di Ingegneria, Università degli Studi di Perugia, Italy

Abstract

A 1-plane graph is a graph embedded in the plane such that each edge is crossed at most once. A 1-plane graph is optimal if it has maximum edge density. A red-blue edge coloring of an optimal 1-plane graph G partitions the edge set of G into blue edges and red edges such that no two blue edges cross each other and no two red edges cross each other. We prove the following: (i) Every optimal 1-plane graph has a red-blue edge coloring such that the blue subgraph is maximal planar while the red subgraph has vertex degree at most four; this bound on the vertex degree is worst-case optimal. (ii) A red-blue edge coloring may not always induce a red forest of bounded vertex degree. Applications of these results to graph augmentation and graph drawing are also discussed.

Keywords: Edge Partitions, 1-Planarity, Optimal 1-Plane Graphs

1. Introduction

A well-studied subject in graph algorithms and graph theory is how to color the edges of a graph such that each partition set induces a subgraph with special properties. For example, Colbourn and Elmallah [20] show how to compute a red-blue edge coloring of a planar graph such that both the red edges and the blue edges form a partial 3-tree. A red-blue edge coloring of a planar graph that partitions the edge set into partial 2-trees is proved by Kedlaya [26] and by Ding *et al.* [17]. Gonçalves [22] shows that every planar graph has a red-blue edge coloring where each color class is an outerplanar graph and that this coloring can be computed in linear time, thus solving a famous conjecture by Chartrand, Geller, and Hedetniemi [11]. As for more than two colors, a classic result by Schnyder [29] is that the edges of a maximal planar graph can be colored in linear time with three colors so that each partition set forms a spanning tree of the graph, which can be used to obtain a quadratic-area planar straight-line drawing.

In this paper we study red-blue edge colorings of graphs that are “almost” planar, that is graphs for which a drawing where only some types of edge crossings are allowed. Namely, we focus on graphs that admit a drawing such that each edge is crossed by at most another edge; these graphs are called *1-planar* and have been the subject of a rich body of literature partly for their applications in the area of graph drawing (see e.g. [2, 3, 4, 6, 9, 10, 13, 16, 18, 19, 21, 24, 28, 32, 33, 34]). A 1-planar graph having maximum edge density is called *optimal 1-planar*. While 1-planar graphs are NP-hard to recognize [23, 27], optimal 1-planar graphs are recognizable in polynomial time [8, 12].

Our research starts with the following observation: It is known that every optimal 1-planar graph with n vertices has exactly $4n - 8$ edges and it admits a drawing with exactly $n - 2$ crossings [13]. Therefore

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*Corresponding author

Email addresses: wlenhart@williams.edu (William J. Lenhart), giuseppe.liotta@unipg.it (Giuseppe Liotta), fabrizio.montecchiani@unipg.it (Fabrizio Montecchiani)

one can assign the red color to one edge per crossing, color all other edges as blue, and obtain two planar subgraphs. The $3n - 6$ blue edges form a maximal planar graph, while the red subgraph consists of $n - 2$ edges. Given the sparsity of the red subgraph, we wonder whether it can have bounded vertex degree, i.e., whether the degree of each vertex in the red subgraph can be bounded by a constant that does not depend on n . For example, Figure 1(a) shows an example where the red subgraph is a forest of two paths: one is $v_{11}, v_2, v_4, v_6, v_9, v_7$; the other path is $v_8, v_5, v_3, v_1, v_{10}, v_{12}$. We recall that Czap and Hudák [13] proved that every optimal 1-planar graph has a red-blue edge coloring such that the blue subgraph is maximal planar and the red subgraph is a forest. Ackerman [1] generalized this result to any (non-optimal) 1-planar graph. However, these results do not give bounds on the vertex degree of the red subgraph.

We prove the following theorem, where by plane (1-plane) graph we mean a planar (1-planar) embedding of the graph in the plane.

Theorem 1. *An optimal 1-plane graph with n vertices admits a red-blue edge coloring such that the blue edges induce a maximal plane graph and the red edges induce a plane graph having vertex degree at most four. This coloring can be computed in $O(n)$ time and the bound on the vertex degree of the red subgraph is worst-case optimal. Also, for any constant upper bound on the vertex degree, the red subgraph may not be a forest.*

As a byproduct of the techniques in the proof of Theorem 1, we prove the following result about planar augmentations of quadrangulations. It may be worth recalling that Corollary 1 is stated as a theorem in [14] where the proof is however omitted, and that the problem of augmenting a planar graph to a triangulated planar graph by minimizing the maximum vertex degree increase is NP-complete [25].

Corollary 1. *Every 3-connected plane quadrangulation with n vertices and maximum vertex degree Δ can be triangulated in $O(n)$ time such that the resulting maximal plane graph has vertex degree at most $\Delta + 4$, and this bound is worst-case optimal.*

A second implication of Theorem 1 concerns visibility representations of non-planar graphs. Visibility representations are a well-known convention to represent graphs and they have been widely studied in the literature, see e.g. [7, 30, 31]. Each vertex is represented as an axis-aligned rectangle and each edge as an unobstructed horizontal or vertical line of sight between the corresponding pair of rectangles. Coloring blue and red the vertical and horizontal edges, respectively, we obtain a red-blue edge coloring of the graph. A visibility representation is *flat* if every rectangle intersects exactly one row [5], which implies that every vertex is incident to at most two red edges. The following corollary follows from Theorem 1.

Corollary 2. *There exist optimal 1-plane graphs not admitting a flat visibility representation.*

The remainder of this paper is organized as follows. Section 2 contains preliminaries. The proof of Theorem 1 is subdivided in two parts. In Section 3 we show that there exist optimal 1-plane graphs not admitting any red-blue edge coloring whose red subgraph is a forest of bounded vertex degree. In Section 4 we prove that every optimal 1-plane graph has a red-blue edge coloring whose red subgraph has vertex degree at most four and this bound is worst-case optimal. In Section 5 we give our final remarks.

2. Preliminaries

Planarity and 1-planarity. A graph G is *simple*, if it contains neither loops nor multiple edges. We consider simple graphs, if not otherwise specified. A *topological graph* is an embedding of a graph in the plane, where the vertices of the graph are distinct points and the edges are Jordan arcs joining the corresponding pairs of points. Also, every pair of arcs share at most one point, which is either a common end-point or an interior point where the two arcs properly cross. A topological graph in which no two edges cross is called a *plane graph*. A graph that admits such a representation is a *planar graph*.

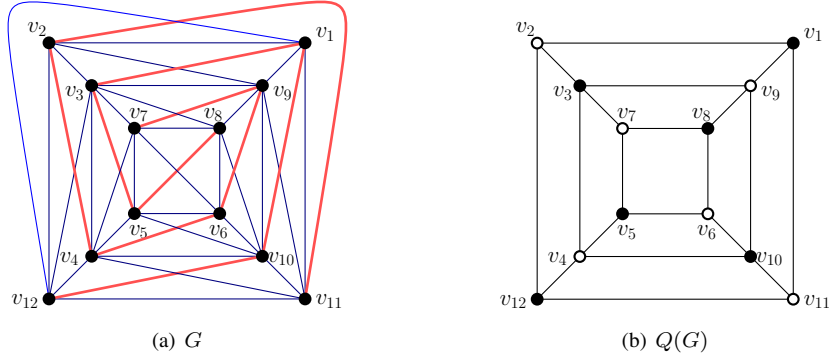


Figure 1: (a) An optimal 1-plane graph G ; the (bold) red edges, one for each crossing, induce two plane paths. (b) The subgraph $Q(G)$ of G , which is a 3-connected quadrangulation.

Furthermore, a topological graph such that every edge is crossed at most once is called a *1-plane graph*, and a graph that admits such a representation is a *1-planar graph*.

A plane graph divides the plane into topologically connected regions, called *faces*. The unbounded region is called the *outer face*. A 1-plane graph still divides the plane into faces, whose boundary may consist of edge segments between vertices and/or crossing points of edges. The unbounded region is still called the outer face.

Every optimal 1-plane graph G can be obtained from a 3-connected plane quadrangulation $Q(G)$ (i.e., a 3-connected plane graph whose faces are all cycles of length four) by adding two (crossed) edges to each face of $Q(G)$ [33]. Since $Q(G)$ is bipartite, this corresponds to coloring the vertices of G with two colors, *black* and *white*, such that the uncrossed edges of G are between vertices with different colors, while all pairs of crossing edges are between vertices of the same color. Based on this fact, we call a crossing edge having black (white) end-vertices a *black (white) diagonal* of G . Figure 1(b) illustrates the subgraph $Q(G)$ of the optimal 1-plane graph G in Figure 1(a).

Book embeddings. A k -page book embedding, for some integer $k \geq 0$, is a particular representation of a graph G , such that: (i) The vertices are restricted to a line, called the *spine*; (ii) The edges are partitioned into k sets, called *pages*, such that edges in a same page are drawn on a half-plane delimited by the spine and do not cross each other.

Let G be an optimal 1-plane graph, and let $Q(G)$ be the plane quadrangulation of G . de Fraysseix *et al.* [15] prove that $Q(G)$ has a 2-page book embedding D such that the following properties hold. We assume the vertices along the spine of D to be ordered from left to right, and we call the two pages of D the *upper* and the *lower* page.

- p1.** The leftmost and the rightmost vertices of D , denoted by s_b and t_b , are both black.
- p2.** The edges of the upper (lower) page induce a spanning tree of $Q(G) \setminus t_b$ ($Q(G) \setminus s_b$).
- p3.** The upper (lower) page contains edges whose leftmost end-vertex is black (white) and whose rightmost end-vertex is white (black).

Figure 3(a) shows a 2-page book embedding which satisfies **p1–p3**.

3. Forest Colorings of Optimal 1-Plane Graphs

A red-blue edge coloring such that the blue subgraph is a maximal plane graph and the red subgraph is a forest is called a *forest coloring*. In this section we show that, for any constant c , there exist optimal 1-

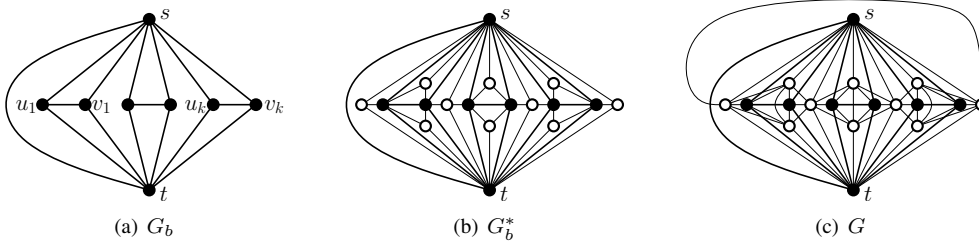


Figure 2: Illustration for the proof of Lemma 2.

plane graphs for which any forest coloring is such that the red subgraph G_R has maximum vertex degree at least c , spans all the vertices of the graph, and is composed of exactly two trees. Hence, G_R has neither bounded vertex degree nor bounded size. We first show that G_R spans all the vertices and is composed of two trees.

Lemma 1. *Let G be an n -vertex optimal 1-plane graph with a forest coloring. Then the red subgraph has n vertices and it is composed of two trees.*

Proof. Denote by n and m the number of vertices and edges of G , respectively. Similarly, denote by n_R and $m_R = n - 2$ the number of vertices and edges of the red subgraph G_R , respectively. Consider the plane quadrangulation $Q(G)$ of G (see Section 2). Recall that the edge set of G_R is composed of one edge for each pair of crossing edges of G , i.e., it contains only black and white diagonals of G . Since G contains at least two white and at least two black vertices, it follows that G_R contains at least two trees, one induced by the white diagonals and one induced by the black diagonals. Let T_1, \dots, T_k be the $k \geq 2$ trees of G_R , each having n_i vertices. Then $m_R = \sum_{i=1}^k (n_i - 1) = \sum_{i=1}^k n_i - k = n - 2$. Since $\sum_{i=1}^k n_i \leq n$ and $k \geq 2$, we have that $\sum_{i=1}^k n_i - k = n - 2$ holds only for $\sum_{i=1}^k n_i = n$ and $k = 2$. \square

To prove the negative result of Theorem 1, we first construct a plane graph G_b which does not admit a spanning tree of bounded degree. We then construct an optimal 1-plane graph G such that G_b is the subgraph of G induced by its black (or, equivalently, white) diagonals. By Lemma 1, we know that the red subgraph of any forest coloring of G spans all the vertices of G and it is composed of exactly two trees. Thus, one of the two trees spans all the black vertices. Moreover, such a tree is a spanning tree of G_b , and thus cannot have bounded degree.

Graph G_b is constructed as follows. Start by adding k vertex-disjoint edges $(u_1, v_1), \dots, (u_k, v_k)$. Connect all the end-vertices of these edges to two additional vertices s and t , and add the edge (s, t) ; see also Figure 2(a). The resulting graph G_b has $n_b = 2k + 2$ vertices, $m_b = 5k + 1$ edges, and, by Euler's formula, $f_b = 3k + 1$ faces. Any spanning tree T_b of G_b must contain either an edge connected to s or an edge connected to t for each of the edges $(u_1, v_1), \dots, (u_k, v_k)$. Thus, either s or t must have degree at least $k/2$ in T_b .

Compute now the so-called *angle graph* G_b^* of G_b by adding a (white) vertex v_f inside each face f of G_b and connect v_f to all vertices on the boundary of f . Graph G_b^* is a triangulated plane graph by construction with $n_b^* = n_b + f_b = 5k + 3$ vertices and $m_b^* = 3n_b^* - 6 = 15k + 3$ edges. Observe that each face of G_b^* contains two black vertices and one white vertex on its boundary; see also Figure 2(b). Finally, for every pair of white vertices, v_f and v'_f , belonging to two faces that share an edge e of G_b , add the edge (v_f, v'_f) and make it cross with e . The resulting graph G is the desired optimal 1-plane graph; see also Figure 2(c). Namely, G is 1-plane by construction, it has $n = n_b^* = 5k + 3$ vertices, and $m = m_b^* + m_b = 15k + 3 + 5k + 1 = 20k + 4 = 4n - 8$ edges (i.e., it is optimal), and its black diagonals are all and only the edges of G_b . We can summarize this discussion as follows.

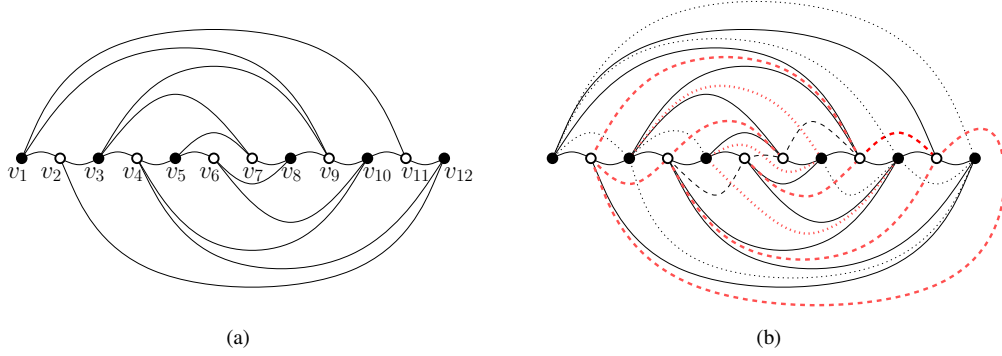


Figure 3: (a) A 2-page book embedding of $Q(G)$ (Figure 1(b)) computed by the algorithm in [15]. (b) A partition of the edges of the optimal 1-plane graph G (Figure 1(a)), computed by algorithm `DiagPicker`. The selected edges are red.

Lemma 2. *For any constant c , there exists an optimal 1-plane graph such that in any forest coloring the maximum vertex degree of the red subgraph is at least c .*

4. k -Degree Colorings of Optimal 1-Plane Graphs

A red-blue edge coloring such that the blue subgraph is a maximal plane graph and the red subgraph has vertex degree at most k (for some constant k) is called a k -degree coloring. In this Section we conclude the proof of Theorem 1 by showing that, if we drop the acyclicity requirement for the red subgraph, then every optimal 1-plane graph G admits a 4-degree coloring. Also, we exhibit a graph not admitting a k -degree coloring for $k < 4$.

We begin by introducing some useful notation. Let G be an optimal 1-plane graph, and let $Q(G)$ be the plane quadrangulation of G . Let D be a 2-page book embedding of $Q(G)$ computed as in [15]. Let f be a face of $Q(G)$, different from the outer face. Due to **p3** of Section 2, the vertices of f along the spine of D can be neither two whites followed by two blacks nor two blacks followed by two whites. Hence, only the following four types of faces are possible in D . Denote by *upper parachute* (resp., *lower parachute*) a face such that its four vertices are an alternated sequence of blacks (resp., whites) and whites (resp., blacks); and denote by *upper dolphin* (resp., *lower dolphin*) a face such that its four vertices are a black (resp., a white) followed by two whites (resp., two blacks) and a black (resp., a white). For example, in Figure 3(a), all inner faces are upper and lower parachutes except for the upper dolphin $\{v_5, v_6, v_7, v_8\}$.

We now describe an algorithm, `DiagPicker`, which exploits the properties of D to select one diagonal for each face of $Q(G)$, so that the graph induced by the selected diagonals has maximum vertex degree four. A red-blue edge coloring such that all selected diagonals are red and all remaining edges are blue yields the desired result.

Step 1. The first step of the algorithm transforms D as follows. Let f be an upper dolphin, and let u and v be the two consecutive white vertices of f (notice that, there is no other vertex between them). Add a (black) vertex w between u and v , and connect w to u and v . This operation splits f into two faces, f_1 and f_2 , such that f_1 is an upper parachute and f_2 is a lower parachute. For lower dolphins we can define an analogous operation. By splitting all upper and lower dolphins in D we obtain a new 2-page book embedding D^* containing only upper and lower parachutes.

Step 2. In the second step, `DiagPicker` selects one diagonal per parachute according to the following rule. We first focus on the upper parachutes, and define some further notation. Let f be an upper

parachute of D^* and let u, v, w, z be its four vertices in the left to right order they appear along the spine. Let f_L be the upper parachute (if any) distinct from f and sharing the edge (u, v) with f . We call f_L the *left child* of f . Similarly, we call the *right child* of f , the upper parachute (if any) f_R sharing the edge (w, z) with f . Furthermore, f is called the *parent* of f_L and f_R .

Let f be an upper parachute. If f has no parent or f is a left child, then the white diagonal of f is selected. Otherwise, if f is a right child, then its black diagonal is selected; see also Figure 3(b) for an example. Let S be the set of selected edges, one for each upper parachute. We prove the following lemma.

Lemma 3. *Let G_S be the graph induced by the edges in S . Then G_S is a forest of paths.*

Proof. To prove the claim, observe first that the parent-child relationship between upper parachutes yields a forest of binary rooted trees, denoted by \mathcal{F} . Namely, if an upper parachute f has f_L and f_R as left and right children, then the corresponding node n_f has n_{f_L} and n_{f_R} as left and right children in a tree of \mathcal{F} . Let T be a tree of \mathcal{F} . Let n_f be a node of T corresponding to an upper parachute f of D^* . Let S_f be the subset of S containing only the selected diagonals for the parachutes in the subtree of T rooted at n_f . We visit T bottom up, and, for each node n_f of T , we show that the following invariant holds: **(I)**. The edges in S_f induce a forest of paths. Let n_f be a leaf of T , then Invariant **(I)** clearly holds.

Claim 1. *Let n_f be a node of T . Let n_{f_L} and n_{f_R} be its two children (one of them may not exist), for which Invariant **(I)** holds. Then Invariant **(I)** holds for n_f .*

Proof. Suppose the white diagonal $e = (v, z)$ of f is selected by `DiagPicker`. Let v be the leftmost end-vertex of e . Vertex v belongs to f , f_L , and to a (possibly empty) set of upper parachutes that correspond to the nodes on the right boundary of T_{f_L} (which are all right children). It follows that v is adjacent to exactly one edge in S_{f_L} (the white diagonal of f_L). Similarly, z belongs to f , f_R , and to a (possibly empty) set of upper parachutes that correspond to the nodes on the right boundary of T_{f_R} . Hence, z is adjacent to no edge in S_{f_R} . Thus, the set $S = S_{f_L} \cup S_{f_R} \cup \{e\}$ satisfies the claim. The argument is symmetric in case `DiagPicker` selected the black diagonal of f . \square

To conclude the proof of the lemma, we observe that each tree of \mathcal{F} is such that the upper parachute corresponding to its root node does not share any vertex with the upper parachutes of any other tree, hence, G_S is a forest of paths. \square

By rotating D^* of an angle $-\pi$, the lower parachutes become upper parachutes, and algorithm `DiagPicker` can be applied so to select one diagonal per upper parachute in the rotated drawing. It follows that, by Lemma 3, the final set S of selected edges (one for each upper and lower parachute), induces a graph G_S of maximum vertex degree four.

Step 3. In the third step, algorithm `DiagPicker` removes the dummy vertices inserted during Step 1, without increasing the maximum vertex degree of the graph G_S . Let f be an upper (resp., lower) dolphin in D , and let f_1 and f_2 be the two parachutes generated by adding a dummy vertex as explained in Step 1. If, for at least one of f_1 and f_2 , a white (resp., black) diagonal e is in G_S , then we can keep e in G_S and remove the other diagonal, which can be a parallel white (resp., black) diagonal between the same pair of vertices or a black (resp., white) diagonal. If for both f_1 and f_2 the black (resp., white) diagonals are in G_S , then we can remove both of them and add the black (resp., white) diagonal of f instead.

We conclude the description of this step by showing a property of G_S that will be used in the last step. Observe that, so far, the algorithm never selected two diagonals which share an end-vertex u and such that the opposite end-vertices are both to the right or both to the left of u . Hence, the following holds.

Property 1. *Let G' be the subgraph of G_S induced by a subset of vertices. Then the leftmost and the rightmost vertices of G' have degree at most two in G' .*

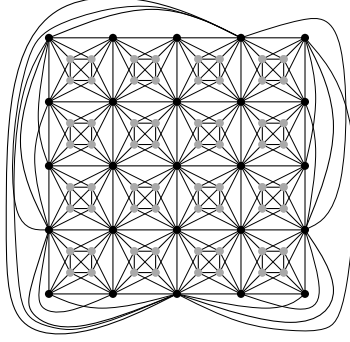


Figure 4: Illustration for the proof of Lemma 5.

Step 4. We conclude by showing how `DiagPicker` chooses one diagonal for the outer face of $Q(G)$ to put in G_S (after this addition Property 1 may not hold anymore). By **p1** of Section 2, the leftmost and the rightmost vertices of D , s_b and t_b , are both black, and they clearly belong to the outer face of $Q(G)$. By Property 1, they both have degree at most two. Thus, we might select the black diagonal that connects s_b and t_b , and the maximum vertex degree of G_S would not exceed four. However, the white diagonal of the outer face of $Q(G)$ can be selected as well. Namely, by **p2** of Section 2, the first vertex after s_b , denoted by v_2 , and the first vertex before t_b , denoted by v_{n-1} , are also on the outer face of $Q(G)$, and thus they are both white. Moreover, v_2 and v_{n-1} have degree at most three, since v_2 (resp., v_{n-1}) has at most two incident edges whose opposite end-vertex is to the left (resp., right) by Property 1, and at most one edge whose opposite end-vertex is to the right (resp., left). Hence, we can select the white diagonal connecting v_2 and v_{n-1} , and the maximum vertex degree of G_S does not exceed four. Figure 3(b) shows the output of `DiagPicker` applied to the graph in Figure 1(a).

We can summarize this discussion as follows.

Lemma 4. *Every optimal 1-plane graph G with n vertices admits a 4-degree coloring that can be computed in $O(n)$ time.*

Proof. We already shown that algorithm `DiagPicker` correctly computes a 4-degree coloring of G . In terms of time complexity, the subgraph $Q(G)$ can be extracted from G in $O(n)$ time by only picking the uncrossed edges. A 2-page book embedding of $Q(G)$ which satisfies **p1–p3** can also be computed in $O(n)$ time [15]. Since $Q(G)$ has $O(n)$ faces, algorithm `DiagPicker` runs in $O(n)$ time. \square

To conclude the proof of Theorem 1, we shall exhibit an optimal 1-plane graph which does not admit a k -degree coloring for $k < 4$. Let $Q(G)$ be the 3-connected plane quadrangulation obtained from the optimal 1-plane graph G in Fig. 4 by removing all pairs of crossing edges. Graph $Q(G)$ has $n = 89$ vertices and, by Euler's formula, $f = n - 2 = 87$ faces. Consider any k -degree coloring of G . The red subgraph G_R contains one diagonal for each face of $Q(G)$, i.e., it has $m_R = 87$ edges. By the handshaking lemma $\sum_{v \in G_R} \deg(v) = 2m_R = 174$. Since each 4-cycle of gray vertices (see Fig. 4) is inside a 4-cycle of black vertices, one can see that the sum of the degrees of each gray 4-cycle in the red subgraph is exactly 6. Also, since the degree of the 25 black vertices in the red subgraph is at most k by assumption, and since we have 16 gray 4-cycles, we have that $\sum_{v \in G_R} \deg(v) \leq 96 + k25$, which implies that $k \geq (174 - 96)/25 > 3$, as desired.

Lemma 5. *There exists an optimal 1-plane graph not admitting any k -degree coloring such that $k < 4$.*

Lemmas 2, 4 and 5 prove Theorem 1.

We recall that any 3-connected plane quadrangulation can serve as the subgraph $Q(G)$ of an optimal 1-plane graph G [33], and that the red edges of a 4-degree coloring of G can be used to triangulate $Q(G)$. We therefore have Corollary 1 as a byproduct of Lemmas 4 and 5.

Finally, Lemma 5 implies Corollary 2.

5. Final Remarks

We proved that it is not possible to pick an edge for each pair of crossing edges of an optimal 1-plane graph G such that the graph induced by the selected edges is a forest of bounded degree. Motivated by this negative result, we showed that if we drop the acyclicity requirement, then an edge decomposition into a triangulated plane graph and a plane graph of maximum vertex degree four can be computed in linear time, and that this bound on the vertex degree is worst-case optimal. It remains open to establish whether every 1-plane graph admits a k -degree coloring for some constant k .

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